J. Fluid Mech. (1970), *vol.* 42, *part* 4, *pp.* 721–731
Printed in Great Britain

On the turbulent flow over a wavy boundary

By RUSS E. DAVIS
Scripps Institution of Oceanography
University of California, San Diego

(Received 11 August 1969 and in revised form 18 February 1970)

Two hypotheses concerning the turbulent flow over an infinitesimal-amplitude travelling wave are investigated. One hypothesis, originally made by Miles, is that the wave does not affect the turbulence and therefore the turbulent Reynolds stresses are dependent only on height above the mean wave surface. Alternatively, the proposal that turbulent stresses are primarily dependent on height above the instantaneous wave surface is examined. Numerical solutions of the appropriate equations are compared with Stewart's recent experimental results and with the approximate solutions employed by Miles and others. No definite conclusion can be reached from comparison with experimental results since the predicted flows are quite sensitive to details of the mean velocity profile near the wave surface where no data was taken. It is found that the asymptotic results do not apply for the conditions investigated.

1. Introduction

Much of the impetus for the recent interest in the mechanism of wind generation of waves came from the pioneering work of Miles (1957). This paper and a subsequent contribution by Benjamin (1959) contained the formulation of the linearized analysis which has served as the basis of most recent investigations of shearing flow over a wavy boundary. Despite a decade of theoretical and experimental investigations, in 1967 Miles found that agreement between theory and experiment had not been achieved. Miles suggested that this was due primarily to the fact that the interaction of the wave and the turbulence was not adequately understood, but, after discussing various *ad hoc* models of the turbulence, he concluded that the experimental data then available did not warrant a detailed investigation of such hypotheses.

Until the dynamics of turbulence is better understood models of flow over waves will, of necessity, be highly speculative. The Miles theory is based on the assumption that the turbulence is not influenced by the wave and consequently the turbulent Reynolds stresses are dependent only on the height above the mean wave surface. This assumption leads to a 'quasi-laminar' wave generation mechanism which is essentially viscous (for a discussion of the limit of zero viscosity see Davis 1969). If, on the other hand, the turbulent Reynolds stresses are influenced by the wave, these stresses may result in a significant change in the flow and a different mechanism for wave generation.

It is not now possible to predict the wave-induced perturbations of the turbulent Reynolds stresses, but one can make plausible assumptions concerning their form, solve the relevant equations of motion and compare the results with experimental data. While such a programme could not be expected to lead to an adequate model of flow over a wave, it would at least determine to what extent the wave-induced velocity field depends on the turbulence and might hopefully allow one to determine which assumed Reynolds stress distribution is most nearly correct. Until recently most experimental work has been concerned with wave growth rates or the pressure distribution on the wave surface. Because there are any number of different flows associated with the same surface pressure or wave growth rate, measurements of these quantities cannot adequately distinguish between different theoretical models. In his recent laboratory study of the flow over surface waves, Stewart (1969) has measured the mean and wave-induced velocity fields over water waves in a wind tunnel. These data provide an appropriate experimental standard with which predicted wave-induced velocity fields can be tested.

In this paper two hypotheses about the wave-induced perturbations of the turbulent Reynolds stresses are examined. Neither hypothesis has any dynamical basis and therefore one must consider them completely unsatisfactory as models of the turbulence. But intercomposition of the predictions based on the two hypotheses does give a measure of the importance of turbulence and comparison with Stewart's data gives some indication of the amount the hypotheses are in error. The first turbulence hypothesis is the 'quasi-laminar' assumption originally introduced by Miles. The second hypothesis is based on Benjamin's (1959) suggestion that, to a first approximation, the properties of the flow are dependent on an appropriate measure of the height above the instantaneous wave surface. While Benjamin did not consider turbulence (as noted in §2, this leads to some confusion when dealing with turbulent mean velocity profiles) his suggestion may be quite plausibly extended to include turbulent quantities.

In order to properly test the two models using experimental data, it is first necessary to solve the appropriate equations without introducing any additional uncertainties arising from mathematical approximations. It is the purpose of this note to present the results of numerical solution of the equations relevant to the two models; these results are compared with Stewart's data and with the approximate solution developed by Miles for the 'quasi-laminar' model.

2. Theoretical models

The essential elements of the Miles–Benjamin model of shearing flow over a wavy boundary are well known (the nomenclature used here is similar to that of Phillips 1966 and Miles 1967) and for our purposes a brief summary will suffice.

All variables are non-dimensionalized using $u^* = (\tau/\rho)^{\frac{1}{2}}$ and $l^* = \nu/u^*$ where τ is the average viscous tangential stress on the boundary, ρ is the fluid density, and ν the kinematic viscosity. Adopting a frame of reference moving in the x direction with speed c the wavy boundary is taken as $z = a\cos kx$. In order to treat the turbulent flow, the span average $\langle\,\rangle$ is defined to be the mean value obtained by averaging over y and/or t while holding z and x constant. The velocity field is decomposed (see Miles 1967) into an overall mean component $\{U(z)+c,\ 0,\ 0\}$, a-wave induced perturbation $\{a\mathscr{U},\ 0,\ a\mathscr{W}\}$ and a turbulent component u' where $\langle u'\rangle = 0$. Similarly, the pressure is written as the sum $P+a\mathscr{P}+p'$.

If terms of $O(a^2)$ are neglected, the equations of motion are

$$a\left\{U\frac{\partial \mathscr{U}}{\partial x}+\mathscr{W}\frac{dU}{dz}+\frac{\partial \mathscr{P}}{dx}-\nabla^2\mathscr{U}\right\} = \frac{d^2U}{dz^2}+\frac{\partial r_{xz}}{\partial z}+\frac{\partial r_{xx}}{\partial x} \equiv X, \tag{1a}$$

$$a\left\{U\frac{\partial \mathscr{W}}{\partial x}+\frac{\partial \mathscr{P}}{\partial z}-\nabla^2\mathscr{W}\right\} = -\frac{dP}{dz}+\frac{\partial r_{zz}}{\partial z}+\frac{\partial r_{xz}}{\partial x} \equiv Z, \tag{1b}$$

where $r_{\alpha\beta} = \langle -u'_\alpha u'_\beta\rangle$ is the turbulent Reynolds stress tensor. Expanding about $z = 0$ the linearized boundary conditions are found to be

$$\mathscr{W} = ikU(0)\,e^{ikx} \quad \text{at} \quad z = 0, \tag{2a}$$

$$\mathscr{U}+e^{ikx} = \mathscr{U}_s \quad \text{at} \quad z = 0, \tag{2b}$$

where the real part of complex quantities is implied and in equation (2b) it has been noted that $U'(0) = 1$. The quantity $U(0)+a\mathscr{U}_s$ is the horizontal velocity of the wave surface; in the absence of any surface drift $U(0) = -c$ and if the water motion corresponds to a free wave $\mathscr{U}_s = cke^{ikx}$. As $z\to\infty$ both $\mathscr{U}$ and $\mathscr{W}$ must vanish.

Taking account of the fact that $\mathscr{U}$ and $\mathscr{W}$ must satisfy the continuity equation they may be expressed as

$$\mathscr{U} = F'(z)\,e^{ikx}, \quad \mathscr{W} = -ikF(z)\,e^{ikx}.$$

Equations (1) can then be combined to

$$ik\{U(F''-k^2F)-U''F\}-F^{\text{iv}}+2k^2F''-k^4F = R(z), \tag{3}$$

where
$$\frac{\partial}{\partial z}\left\{\frac{d^2U}{dz^2}+\frac{\partial r_{xz}}{\partial z}+\frac{\partial r_{xx}}{\partial x}\right\}-\frac{\partial}{\partial x}\left\{\frac{r_{zz}}{\partial z}+\frac{\partial r_{xz}}{\partial x}\right\} = aR(z)\,e^{ikx}.$$

For this analysis to apply, the total shear stress must be approximately independent of z, that is $\partial(U'+r_{xz})/\partial z = O(a)$.

The 'quasi-laminar' model of flow over a wave results from the assumption that the turbulent Reynolds stresses are functions of z, the height above the mean position of the wave. In this case the inhomogeneous terms X, Z and R are zero, and mathematically the problem takes a form identical to the problem of a laminar flow over a wavy boundary.

An equally plausible hypothesis is that the Reynolds stresses are, to a first approximation, dependent on an appropriate measure of the distance above the instantaneous water surface rather than on the height above the mean surface $z = 0$. Consequently, we shall investigate the proposition that the Reynolds stresses are constant along lines of constant $\eta = z-a\,e^{-kz+ikx}$. The overall mean stress is independent of z so that

$$\bar{r}_{xz}+U' = \tau_0,$$

where $\bar{r}_{xz}$ is the overall average of r_{xz} and τ_0 is a constant. From the hypothesis that $r_{xz} = r_{xz}(\eta)$ it follows that

$$r_{xz} = \tau_0-U'+U''a\,e^{-kz+ikx}.$$

The normal stresses r_{zz} and r_{xx} are not easily determined, but experimental evidence (for example, see Hinze 1959, ch. 7) suggests that $\bar{r}_{xx} = \bar{r}_{zz} = 2{\cdot}4\bar{r}_{xz}$ is a reasonable approximation. From the hypothesis that the Reynolds stresses are functions of η it follows that $r_{xx} = r_{zz} = 2{\cdot}4 r_{xz}$. While there is considerable uncertainty about the assumed form of $\bar{r}_{xx}$ and $\bar{r}_{zz}$ the influence of the normal Reynolds stresses on the final result is unimportant.

The above expressions for the turbulent Reynolds stresses may now be used to determine the inhomogeneous terms in equations (1) and (3);

$$X = a\{U''' + (2{\cdot}4i - 1)kU''\}e^{-kz+ikx}, \tag{4a}$$

$$Z = a\{2{\cdot}4U''' + (i - 2{\cdot}4)kU''\}e^{-kz+ikx}, \tag{4b}$$

$$R = (U^{\text{iv}} - 2kU''' + 2k^2U'')\,e^{-kz}. \tag{4c}$$

It is coincidental that the terms X, Z and R are similar to the inhomogeneous term appearing in equation (3.1) of Benjamin's (1959) formulation of the 'quasi-laminar' model. Why these inhomogeneous terms arise in Benjamin's formulation and not in Miles's is a point deserving some clarification.

Suppose that the turbulent quantities are completely neglected in equations (1) and no account is taken of the fact that, in the absence of turbulent stresses, only a linear mean profile can be truly parallel. If these equations are transformed to the curvilinear co-ordinates

$$\begin{Bmatrix}\eta\\ \xi\end{Bmatrix} = \begin{Bmatrix}z\\ x\end{Bmatrix} - a\begin{Bmatrix}i\\ 1\end{Bmatrix}e^{-kz+ikx}$$

and terms of $O(1)$ are neglected, the resulting inhomogeneous equations are those given by Benjamin. If the x, z co-ordinate system is retained and terms of $O(1)$ are neglected, the equations are homogeneous. The velocity fields derived from these seemingly identical analyses are not identical.

The origin of this inconsistency is, as Benjamin pointed out, the fact that the mean flow is not a solution of the equations of motion. The inhomogeneous term which appears in one derivation and does not appear in the other is simply the algebraic difference between the $O(1)$ quantities which are neglected in the two developments. In deriving the vorticity equation in Cartesian co-ordinates the term d^3U/dz^3 is neglected but if this quantity is expressed as a function of ξ and η it is seen that it differs by $O(a)$ from the quantity which is neglected in Benjamin's derivation. If those terms which are formally $O(1)$ are sufficiently small to be neglected then the inhomogeneous term in the perturbation equation must also be neglected. On the other hand, if these terms are large (as is the case for turbulent velocity profiles) then the analysis cannot be rationally carried to $O(a)$ without including the additional quantities which balance the profile curvature. If, for example, the Reynolds stresses are retained then the results obtained using either Cartesian or curvilinear co-ordinates are identical; the range of validity of the linearization is, however, greater for the formulation employing curvilinear co-ordinates.

3. Numerical technique

The perturbation velocities $\mathscr{U}$ and $\mathscr{W}$ are determined by $F(z)$, the solution to equation (3), and the appropriate boundary conditions at $z = 0$ and $z \to \infty$.

As discussed earlier it was deemed essential to obtain accurate numerical solutions of these equations in order to avoid the uncertainties introduced by the simplifications used in obtaining approximate analytic solutions. It was initially felt that accurate solutions could be obtained by approximating the two-point boundary-value problem by a single system of algebraic equations. Unfortunately, the memory size of the available computer (a Control Data Corporation 3600) was too small to allow the use of a finite-difference interval small enough to demonstrate convergence.

Recourse was then made to a modification of the integration scheme developed by Kaplan (1964). This technique attempts to find linearly independent solutions of equation (3) by integrating out from the end-points using different starting conditions. While this technique has modest storage requirements, it is subject to a difficulty associated with the fact that the two 'viscous' solutions have much larger growth rates than the relatively well-behaved 'inviscid' solutions. If the integration is begun with starting conditions corresponding to a linear combination of a rapidly growing solution F_R and a slowly growing function F_S then, before the integration has proceeded far, F_R will dominate and F_S will be lost because of round-off errors. Because the appropriate starting condition for F_S cannot be determined *a priori*, it is necessary to remove F_R as the integration progresses.

Only those solutions which vanish as $z \to \infty$ are required. The 'viscous' solution, F_R, is found by integrating in from $z = 1500$ using a fourth-order Runge–Kutta algorithm and starting conditions derived from the solution of (3) for $U =$ constant. A second solution is started at $z = 1500$ using initial conditions derived from the inviscid form of (3). This solution is extended using the following scheme: (*a*) integrate a short distance to find $F_n = F_s + AF_R$ where A is, of course, unknown; (*b*) obtain an estimate, A_n, of the unknown A and define the new solution $F_{n+1} = F_n - A_n F_R$; (*c*) extend F_{n+1} as in step (*a*) and repeat the filtering of step (*b*) until $z = 0$ is reached. It was found that choosing A_n such that $F''_{n+1} - k^2 F_{n+1} = 0$ was satisfactory to guarantee that the second solution was linearly independent from F_R.

The simultaneous equation method could be used to construct convergent solutions to this problem if the outer boundary conditions were placed at $z = 200$. These solutions were used to test the filtering scheme. In addition, it was verified that altering the step size and filtering interval by a factor of three produced a variation in the solution, F, which was everywhere less than 0·1 per cent of F or 10^{-6}, whichever was greater. In addition, a mean velocity profile similar to that employed by Reynolds (private communication), who calculated solutions of the 'quasi-laminar' model, was used and the result agreed within a few per cent; the discrepancy was ascribed to the difference between the profiles.

4. Results

The numerical technique described above was used to investigate two basic questions: (*a*) Can Stewart's data be used to indicate which of the turbulence hypotheses discussed in §2 is most appropriate? (*b*) How do the flows predicted by the two different forms of equation (3) compare with each other and with the asymptotic solutions employed by Miles (1957, 1959) and Benjamin (1959)?

In order to properly test the two turbulence hypotheses using Stewart's data, it was first necessary to determine how much the theoretical predictions were affected by factors which were not measured or which were subject to some experimental error. Thus for each of the experimental conditions, equation (3) was integrated numerically using various different mean profiles which were consistent with the measured profile but which differed from each other near the wave surface where no measurements were made. Similarly, the influence of a small change in the friction velocity, u^*, was investigated. Further the effects of a small surface drift velocity ($U(0) = 0{\cdot}5 - c$) and an oscillating tangential surface velocity ($\mathscr{U}_s = ck\exp(ikx)$) were determined to be negligible.

In his paper Stewart (1969) has presented several typical theoretical wave-induced velocity fields along with his experimental data. In that paper the hypothesis that the turbulent Reynolds stresses are functions of z is referred to as model A and the hypothesis that the stresses are functions of $\eta = z - a\exp(-kz + ikx)$ is referred to as model B.

From the results presented in Stewart's paper, certain qualitative features are evident. The small changes in mean profile produced changes in the predicted wave-induced velocities as large as the differences between the two different turbulent Reynolds stress distributions; the 'quasi-laminar' model is more sensitive to the mean profile than is the 'turbulent' model. At high wind speeds (low c) the predicted magnitudes of $\mathscr{U}$ and $\mathscr{W}$ are generally larger than the measured values whereas at low wind speeds the converse is often the case; the 'quasi-laminar' model generally yields larger velocity fluctuations than does the hypothesis $r = r(\eta)$. Neither hypothetical Reynolds stress compares very well with the data although the assumption that the Reynolds stress tensor is a function of η appears to be more satisfactory for the higher wind speeds.

It is interesting to compare the numerical results with the approximate analytic solutions employed by Miles (1957) and Benjamin (1959) in their investigations of the 'quasi-laminar' model. According to these solutions, which are formally valid in the limit of infinite Reynolds number, viscous effects are confined to thin layers near the wave surface and near the critical height, z_c, where $U = 0$. Asymptotic solutions may be obtained either when these two layers are well separated or when both are contained within a region over which the mean profile may be approximated by a linear relation. None of the cases investigated numerically fall into this latter category.

As Lin (1955) has shown, where viscous effects are negligible the wave induced Reynolds stress $\sigma \equiv -\overline{\mathscr{U}\mathscr{W}}$ is constant. Above the critical layer $\sigma = 0$ and if the two viscous layers do not overlap $\sigma = -\pi k^{-1}\overline{\mathscr{W}}_c^2 U_c''/U_c'$ in the region between the critical and boundary layers.

The only numerical results which showed a region of constant wave-induced Reynolds stress below the critical height were those corresponding to the lowest wind speed (largest c) for which there was a critical layer. Figure 1 is a plot of σ for one such case; for this integration $c = 16{\cdot}4$, $k = 0{\cdot}005$ and $z_c = 155$. For $70 < z < 150$, σ is approximately equal to the value predicted by the asymptotic theory but since σ varies significantly over the range $0 < z < 70$ it is clear that the viscous wall layer is much thicker than might be expected from the asymptotic theory.

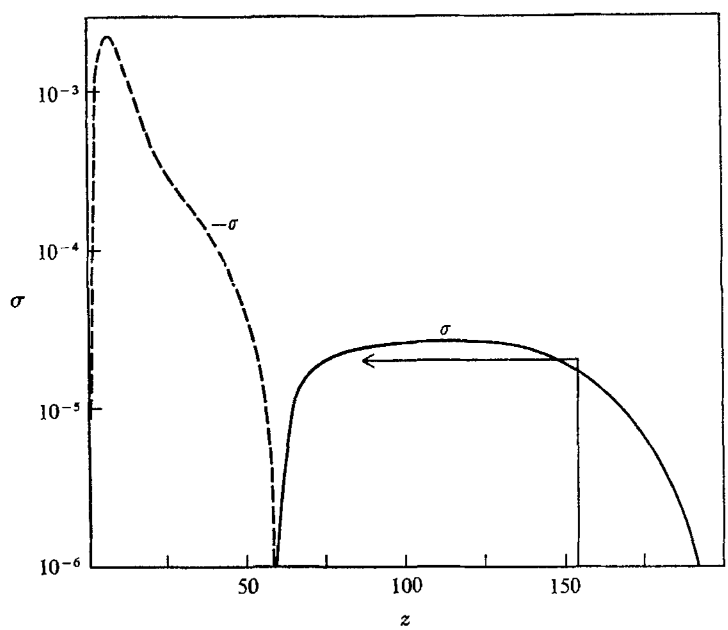

FIGURE 1. The Reynolds stress $\sigma = -\overline{\mathscr{U}\mathscr{W}}$ *vs.* z for the 'quasi-laminar' model with $c = 16{\cdot}4$ and $k = 0{\cdot}005$. The dashed line represents $\sigma < 0$ and the horizontal line the value of σ predicted by the asymptotic solution employed by Miles.

An examination of the inviscid form of equation (3) demonstrates that in regions where viscous stresses are negligible F will vary approximately as U itself. The function F may be expressed as the sum of F_I, the solution of the inviscid equation, and a viscous correction F_V, which for a semi-logarithmic U is $O(F_I k^{-1} z^{-2})$. Thus, if $z^2 k \gg 1$ then F will be well approximated by F_I; this, however, does not mean that the wave-induced Reynolds stress, σ, will be constant. From the definition of σ it follows that

$$2\sigma/k = \operatorname{Im}\{F'F^*\} \approx \operatorname{Im}\{F_I'F_I^* + F_I'F_V^* + F_V'F_I^*\}, \tag{5}$$

where * denotes the complex conjugate. The first term on the right-hand side of (5) represents the Reynolds stress carried by the solution to the inviscid equation; it is necessarily constant and from the asymptotic solution is expected to be of $O(|F_c|^2 z_c^{-1})$. The remaining terms are viscous corrections which are not constant and are of $O(|F|^2 k^{-1} z^{-3})$. Consequently a constant wave-induced Reynolds stress is to be expected only if

$$\gamma \equiv \frac{kz^3}{z_c} \frac{|F_c|^2}{|F|^2} \gg 1.$$

From the numerical results leading to the plot of σ given in figure 1, it was found that $\gamma = 1$ at $z = 45$ and $\gamma = 13$ at $z = 70$, the point where σ ceased to vary rapidly. This tends to confirm the criteria that γ must be large if the wave-induced Reynolds stress is to be constant.

One of the energy inputs for wave generation is the component of pressure which is in phase with the surface slope. Adopting the Miles (1957, 1959) notation, we take the pressure at the wave surface to be

$$P_s - P_0 = \rho(2{\cdot}5u^*)^2 ak \,\mathrm{Re}\,(\alpha + i\beta)\, e^{ikx}.$$

The quantity β is of primary interest so far as wave generation is concerned.

c	7·2	10·1	12·5	14·5	15·2	16·4	24·9
k	0·0022	0·0031	0·0039	0·0045	0·0047	0·0051	0·0077
				$\delta = 7$			
β_1	3·9	2·2	1·0	0·6	0·6	0·3	0
β_2	2·4	−6·5	−1·6	0·5	0·7	0·6	0·7
				$\delta = 10$			
β_1	5·2	2·3	1·3	0·6	0·4	0·2	0
β_2	−4·0	−13·7	1·8	1·0	0·3	0·6	0·6
				$\delta = 15$			
β_1	8·6	2·9	1·7	0·7	0·4	0·3	0
β_2	−23·2	−12·7	6·2	0·8	0·4	0·6	0·7

TABLE 1. The pressure parameter β. Subscript 1 denotes the 'turbulent' model while 2 denotes the 'quasi-laminar' model. The parameter δ refers to the mean velocity profiles shown in figure 4 of Stewart's (1969) paper

Values of β have been computed for both turbulence hypotheses through integration of equation (1*b*). Table 1 contains the value of β found for both turbulence models using each of the three mean velocity profiles shown in figure 4 of Stewart's (1969) paper. The fact that the 'quasi-laminar' model yields large negative values of β for the largest wind speed is surprising. It should be emphasized, however, that contrary to intuition viscosity becomes more important as the wind speed increases; the cases which result in negative values of β do not correspond to the large Reynolds number conditions investigated in the asymptotic theories. For the lower wind speeds both models yield values of β which are of the same magnitude as those given by Miles (1959).

It is a basic tenet of the Miles wave-generation mechanism that the momentum transferred to the wave is extracted from the mean flow in the critical layer and carried downwards by the Reynolds stress $\sigma = -\overline{\mathscr{U}\mathscr{W}}$. The numerical results for $c = 16{\cdot}4$ appear to be consistent with this mechanism in that there is a significant range below the critical level over which viscous stresses are negligible and σ is approximately constant (see figure 1). However, for the three mean velocity profiles considered, the wave-induced Reynolds stress accounted for only about half of the momentum carried to the wave surface by the pressure in phase with the wave slope. It is apparent, therefore, that for this particular choice of c and k, viscous effects near the wave surface are more important than envisioned in the Miles theory.

In addition to the work done by pressure, tangential stresses in phase with the tangential surface velocity at the wave surface can lead to wave growth (cf. Longuet-Higgins 1969). So far as growth of a nearly irrotational wave is concerned, a fluctuating tangential stress $\tau \exp(ikx)$ results in the same wave growth as the normal stress $i\tau \exp(ikx)$. Thus a tangential stress in phase with wave elevation is dynamically equivalent to a normal stress in phase with the wave slope.

c	7·2	10·1	12·5	14·5	15·2	16·4	24·9
k	0·0022	0·0031	0·0039	0·0045	0·0047	0·0051	0·0077
				$\delta = 7$			
ϕ_1	1·0	0·4	0·2	0·1	0	−0·1	−1·2
ϕ_2	2·2	2·4	2·2	1·9	1·8	1·6	0
				$\delta = 10$			
ϕ_1	0·1	−0·2	−0·2	−0·3	−0·4	−0·5	−1·5
ϕ_2	1·8	2·7	2·4	2·1	1·9	1·7	−0·1
				$\delta = 15$			
ϕ_1	0·2	−0·1	−0·2	−0·2	−0·3	−0·4	−1·5
ϕ_2	1·3	3·3	2·8	2·5	2·3	2·0	0·1

TABLE 2. The tangential stress parameter ϕ. Subscript 1 denotes the 'turbulent' model while 2 denotes the 'quasi-laminar' model. The parameter δ refers to the mean velocity profiles shown in figure 4 of Stewart's (1969) paper.

The total tangential stress on the surface is

$$\tau - \tau_0 = a\left\{U'' + \frac{d\overline{r_{xz}}}{dz}\right\}_{z=0} e^{ikx} + \left[a\left\{\frac{\partial \mathscr{U}}{\partial z} + \frac{\partial \mathscr{W}}{\partial x}\right\} + r_{xz} - \bar{r}_{xz}\right]_{z=0}.$$

Since $\bar{r}_{xz} = (\tau_0 - U')$ only the second term contributes to the tangential stress. For an irrotational deep water wave the boundary condition (2b) is

$$\mathscr{U} = (-1 + ck)e^{ikx} \quad \text{at} \quad z = 0. \tag{6}$$

Except in the unusual case $ck > 1$ the wave-induced velocity at the wave crest is directed opposite the propagation velocity. This may be expected to make $\partial \mathscr{U}/\partial z$ positive at the crest, a situation which is favourable to wave growth.

Numerical results obtained using the boundary condition (6) have been used to compute values of ϕ defined by the relation

$$T_s - T_0 = \rho(2{\cdot}5u^*)^2 ak \,\mathrm{Re}\,(\phi + i\theta)e^{ikx},$$

where T_s and T_0 are the dimensional forms of τ_s and τ_0 respectively. The parameter ϕ is a measure of wave growth similar to β so that equal values of β and ϕ indicate equal contributions to wave growth. Table 2 contains the value of ϕ

obtained numerically for each of the two turbulence hypotheses discussed in §2. It is evident that the tangential stresses are most favourable for high wind speeds (low c). The 'quasi-laminar' turbulence model yields values of ϕ corresponding to a energy flux which is appreciable compared with the work done by pressure. The alternate turbulence hypothesis generally results in energy being extracted from the wave by the tangential stress.

A third source of energy for the wave is the viscous normal stress which is in phase with the wave slope. The viscous stress acting on the wave surface leads to a correction to β which, for an irrotational wave, is $(-2+ck)/6{\cdot}25$. Except in the unlikely case $ck > 2$, this correction is unfavourable to wave growth. For the lower wind speeds investigated numerically this effect is comparable to the work done on the wave by pressure.

5. Conclusion

Neither of the turbulence models investigated here was arrived at through consideration of the dynamics of turbulence and consequently it is not surprising that neither model results in good agreement with the data. Some recent experiments by Kendall (private communication) show that there are significant wave-induced fluctuations in the turbulent Reynolds stress (in contrast to the 'laminar' model) and that they are not in phase with the wave elevation (in contrast to the 'turbulent' model). The considerable differences between the wave-induced velocity fields predicted by these two models demonstrates that the problem of wave generation cannot be understood until the wave's influence on the turbulent stresses is known.

Comparison of the numerical solutions of the 'quasi-laminar' model and the asymptotic solutions developed by Miles (1957) and others demonstrates that the effects of viscosity can be important even under conditions when the asymptotic solutions seem to pertain. In order to accurately delineate the region of validity of the asymptotic results, a systematic numerical investigation is required.

At this point it can, however, be said that the Miles (1957, 1959) theory will not be accurate unless the critical layer is outside the viscous wall layer which, for a semi-logarithmic mean profile, requires

$$kz_c^2 \gg 1 \quad \text{or} \quad \frac{h_c u^*}{\nu} \gg \left(\frac{\lambda}{h_c}\right),$$

where h_c is the dimensional critical height and λ is the wavelength. Further, Davis (1969) has shown that the model is invalid if

$$\frac{h_c}{a} \ll \left(\frac{h_c u^*}{\nu}\right)^{\frac{2}{3}},$$

where a is the dimensional wave amplitude. These considerations seriously limit the conditions for which the theory is expected to apply.

The author wishes to thank Professor J. W. Miles and Dr R. H. Stewart for their co-operation. This work was completed under contract NONR 2216(23).

REFERENCES

BENJAMIN, T. B. 1959 Shearing flow over a wavy boundary. *J. Fluid Mech.* **6**, 161.

DAVIS, R. E. 1969 On the high Reynolds number flow over a wavy boundary. *J. Fluid Mech.* **36**, 337.

HINZE, J. O. 1959 *Turbulence.* New York: McGraw-Hill.

KAPLAN, R. E. 1964 Solution of the Orr–Sommerfeld equation for laminar boundary layer flow over compliant boundaries. *Cambridge Aeroelastic and Structures Research Laboratory* ASRL–TR–116–1, *Mass. Inst. Tech.*

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability.* Cambridge University Press.

LONGUET-HIGGINS, M. S. 1969 On the action of a variable stress at the surface of water waves. *Phys. Fluids*, **12**, 737.

MILES, J. W. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185.

MILES, J. W. 1959 On the generation of surface waves by shear flows, Part 2. *J. Fluid Mech.* **6**, 568.

MILES, J. W. 1967 On the generation of surface waves by shear flows, Part 5. *J. Fluid Mech.* **30**, 163.

PHILLIPS, O. M. 1966 *The Dynamics of the Upper Ocean.* Cambridge University Press.

STEWART, R. H. 1969 Laboratory studies of the velocity field over water waves. *J. Fluid Mech.* **42**, 733.